From the finite to the transfinite: $\Lambda\mu$-terms and streams

WIR 2014

Fanny He
f.he@bath.ac.uk

Alexis Saurin
alexis.saurin@pps.univ-paris-diderot.fr

UNIVERSITY OF BATH

université PARIS DIDEROT PARIS 7

12 July 2014

The $\Lambda\mu$-calculus

Syntax of $\Lambda\mu$

$$t ::= x \mid \lambda x.t \mid (t)u \mid \mu\alpha.t \mid (t)\alpha$$

λ-variables: $x \dots$, μ-variables: $\alpha, \beta \dots$

- Type system for $\Lambda\mu$:

$$\frac{}{\Gamma, x : A \vdash x \;: A|\Delta}\ Var \qquad \frac{\Gamma, x : A \vdash t \;: B|\Delta}{\Gamma \vdash \lambda x^A.t \;: A \to B|\Delta}\ \lambda Abs$$

$$\frac{\Gamma \vdash t \;: A \to B|\Delta \qquad \Gamma \vdash u \;: A|\Delta}{\Gamma \vdash (t)u \;: B|\Delta}\ \lambda App$$

$$\frac{\Gamma \vdash t \;: \bot|\Delta, \alpha : A}{\Gamma \vdash \mu\alpha^A.t \;: A|\Delta}\ \mu Abs \qquad \frac{\Gamma \vdash t \;: A|\Delta, \alpha : A}{\Gamma \vdash (t)\alpha \;: \bot|\Delta, \alpha : A}\ \mu App$$

$$\frac{}{\Gamma, x : A \vdash x \ : A|\Delta}\ Var \qquad \frac{\Gamma, x : A \vdash t \ : B|\Delta}{\Gamma \vdash \lambda x^A.t \ : A \rightarrow B|\Delta}\ \lambda Abs$$

$$\frac{\Gamma \vdash t \ : A \rightarrow B|\Delta \qquad \Gamma \vdash u \ : A|\Delta}{\Gamma \vdash (t)u \ : B|\Delta}\ \lambda App$$

$$\frac{\Gamma \vdash t \ : \bot|\Delta, \alpha : A}{\Gamma \vdash \mu\alpha^A.t \ : A|\Delta}\ \mu Abs \qquad \frac{\Gamma \vdash t \ : A|\Delta, \alpha : A}{\Gamma \vdash (t)\alpha \ : \bot|\Delta, \alpha : A}\ \mu App$$

Example with Peirce's law:

$$\vdash \lambda x.\mu\alpha.((x)\lambda y.\mu\beta.(y)\alpha)\alpha : ((A \longrightarrow B) \longrightarrow A) \longrightarrow A$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : (A \rightarrow B) \rightarrow A \vdash x \ : (A \rightarrow B) \rightarrow A|}\ Var \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{y : A \vdash y \ : A|}\ Var}{y : A \vdash (y)\alpha \ : \bot|\alpha : A}\ \mu App}{y : A \vdash \mu\beta.(y)\alpha \ : B|\alpha : A}\ \mu Abs}{\vdash \lambda y.\mu\beta.(y)\alpha \ : A \rightarrow B|\alpha : A}\ \lambda Abs}{x : (A \rightarrow B) \rightarrow A \vdash (x)\lambda y.\mu\beta.(y)\alpha \ : A|\alpha : A}\ \lambda App}{x : (A \rightarrow B) \rightarrow A \vdash ((x)\lambda y.\mu\beta.(y)\alpha)\alpha \ : \bot|\alpha : A}\ \mu App}{\dfrac{x : (A \rightarrow B) \rightarrow A \vdash \mu\alpha.((x)\lambda y.\mu\beta.(y)\alpha)\alpha \ : A|}{\vdash \texttt{call/cc} \ : ((A \rightarrow B) \rightarrow A) \rightarrow A|}\ \lambda Abs}\ \mu Abs$$

$\longrightarrow_{\mathsf{fst}}$-reduction

An example with the reduction $\longrightarrow_{\mathsf{fst}}$:

$$\mu\alpha \,.\, z \longrightarrow_1 \lambda x_1 \mu\alpha \,.\, z$$

$\longrightarrow_{\mathsf{fst}}$-reduction

An example with the reduction $\longrightarrow_{\mathsf{fst}}$:

$$\mu\alpha \, . \, z \longrightarrow_2 \lambda x_1 \lambda x_2 \mu\alpha \, . \, z$$

$\longrightarrow_{\mathsf{fst}}$-reduction

An example with the reduction $\longrightarrow_{\mathsf{fst}}$:

$$\mu\alpha\,.\,z \longrightarrow_3 \lambda x_1 \lambda x_2 \lambda x_3 \mu\alpha\,.\,z \text{ and so on...}$$

$\longrightarrow_{\mathsf{fst}}$-reduction

$$\boxed{\mu\alpha.z \ \longrightarrow^{n}_{\mathsf{fst}} \ \lambda x_1 \ldots x_n.\mu\alpha.z \qquad x_1, \ldots, x_n \neq z\,,\ n \in \omega}$$

$$\boxed{\mu\alpha.t \longrightarrow_{\mathsf{fst}} \lambda x.\mu\beta.t\{(u)x\beta/(u)\alpha\}}$$

$$\mu\alpha \sim \lambda x_1 x_2 \ldots$$

Outline

Böhm trees for the $\Lambda\mu$-calculus

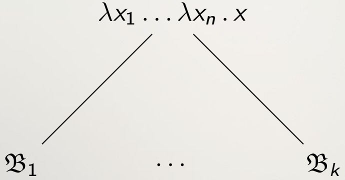
Transfinite calculi and $\Lambda\mu$

Current directions

Böhm trees for λ-calculus

Let $t \in \Lambda$, then t can be written:
$\lambda \vec{x_0} \,.\, (t_0)\, \vec{t_1}$ where t_0 variable or redex :

- If t has no hnf, $\mathfrak{B}_t = \Omega$
- If $t \longrightarrow_* \lambda x_1 \ldots x_n.(x)t_1 \ldots t_k$,
  and $\mathfrak{B}_1, \ldots, \mathfrak{B}_k$ Böhm trees of $t_1, \ldots, t_k$
  $\mathfrak{B}_t =$



- Böhm trees $\mathfrak{B}$ for λ-calculus :

$$\boxed{\mathfrak{B} ::= \Omega \mid \lambda (x_i)_{i \in \nu}.(y)(\mathfrak{B}_j)_{j \in \gamma}} \quad \nu, \gamma \in \omega$$

Böhm trees for λ-calculus

- Any term can be written:
  $\lambda \vec{x_0} \,.\, (t_0)\, \vec{t_1}$ where t_0 variable or redex
- Böhm trees $\mathfrak{B}$ for λ-calculus :

$$\boxed{\mathfrak{B} ::= \Omega \mid \lambda (x_i)_{i \in \nu}.(y)(\mathfrak{B}_j)_{j \in \gamma}} \quad \nu, \gamma \in \omega$$

Böhm trees for $\Lambda\mu$

- Any $\Lambda\mu$-term can be written:
  $\lambda \vec{x_0} \mu\alpha_0 \ldots \mu\alpha_n \lambda \vec{x_{n+1}}.(t_0)\, \vec{t_1} \beta_1 \ldots \beta_m \vec{t_{m+1}}$, where t_0 variable or "pre-redex"
- Böhm trees for $\Lambda\mu \mathfrak{B} \in \Lambda\mu$-$\mathfrak{BT}$:

$$\boxed{\mathfrak{B} ::= \Omega \mid \lambda (x_i)_{i \in \nu}.(y)(\mathfrak{B}_j)_{j \in \gamma}} \quad \nu, \gamma \in \omega^2$$

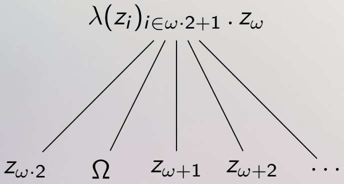
An example

Let $t = \mu\alpha.\lambda x.\mu\beta.\lambda y.((x)y\ ((\Delta)\Delta)\beta)\ \beta$.

Intuition: $t \sim \underbrace{\lambda x_1^\alpha \ldots}_{\omega}\ \lambda x.\ \underbrace{\lambda x_1^\beta \ldots}_{\omega}\ \lambda y.((x)y\ ((\Delta)\Delta)\underbrace{x_1^\beta \ldots)}_{\omega}\ \underbrace{x_1^\beta \ldots}_{\omega}$

$\mathfrak{B} = \lambda(z_i)_{i \in \omega \cdot 2+1}.(z_\omega)(\mathfrak{B}_j)_{j \in \omega}$ with

- $\mathfrak{B}_0 = z_{\omega \cdot 2}$,
- $\mathfrak{B}_1 = \Omega$ and
- $\mathfrak{B}_{j+1} = z_{\omega + j}$ for $1 \leq j < \omega$.



Outline

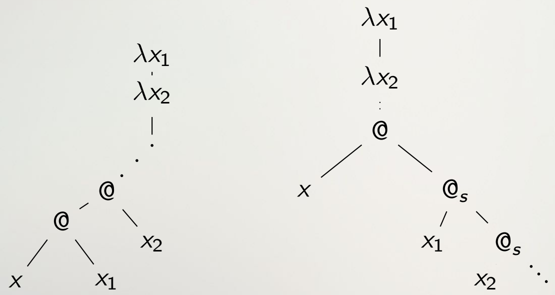
Böhm trees for the $\Lambda\mu$-calculus

Transfinite calculi and $\Lambda\mu$

Current directions

Infinite terms from $\Lambda\mu$-terms

- Limits at root-positions: infinitary representations of $\mu\alpha.(x)\alpha$:



- To solve this issue we introduce a constructor for streams:

$$\begin{array}{ll} \text{Terms} & t ::= x \mid \lambda x.t \mid (t)u \mid \mu\alpha.t \mid (t)S \\ \text{Streams} & S ::= \alpha \mid [t|S] \end{array}$$

$$\mu\alpha.t \longrightarrow_{\mathsf{fst}} \lambda x.\mu\beta.(t\{[x \mid \beta]/\alpha\})$$

- The resulting terms are not infinitary terms since they have subterms at infinite depth: See Ketema et al.

Outline

Böhm trees for the $\Lambda\mu$-calculus

Transfinite calculi and $\Lambda\mu$

Current directions

Ketema *et al.* transfinite terms

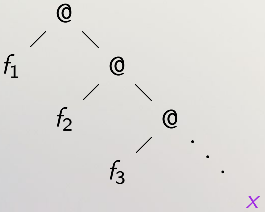
- Transfinite (Tr.) position $p : length(p) \longrightarrow \mathbb{N}$,
- Tr. term $t : \mathcal{P} \longrightarrow \Sigma \cup X$, $\mathcal{P}$ set of positions
  - $p \in \mathcal{P} \longrightarrow \forall q < p,\ q \in \mathcal{P}$,
  - $t(p)$ function symbol of arity $n \Longrightarrow (p \cdot i \in \mathcal{P}$ iff $1 \leq i \leq n)$,
  - (p has limit ordinal length and $\forall q < p,\ q \in \mathcal{P}) \Longrightarrow p \in \mathcal{P}$

Ketema *et al.* transfinite terms

- Transfinite (Tr.) position $p : length(p) \longrightarrow \mathbb{N}$,
- Tr. term $t : \mathcal{P} \longrightarrow \Sigma \cup X$, $\mathcal{P}$ set of positions
  - $p \in \mathcal{P} \longrightarrow \forall q < p,\ q \in \mathcal{P}$,
  - $t(p)$ function symbol of arity $n \Longrightarrow (p \cdot i \in \mathcal{P}$ iff $1 \leq i \leq n)$,
  - $(p$ has limit ordinal length and $\forall q < p,\ q \in \mathcal{P}) \Longrightarrow p \in \mathcal{P}$
- Tr. Term Rewriting System (tTRS): pair (Σ', R), where Σ' set of symbols of finite arity, R set of tr. rewrite rules,
- Tr. rewrite steps: rewriting a tr. term $s = C[\sigma(l)]$ into $t = C[\sigma(r)]$ if $l \to r \in R$, $C[\square]$ one-hole context and σ substitution.

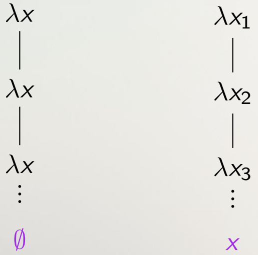
Intuition behind Ketema *et al.* transfinite terms

Two representations for the "transfinite" term $(f_1(f_2(f_3(\ldots x))))$:

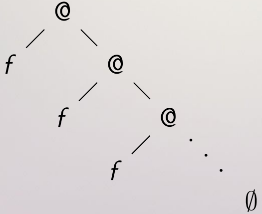


Contrasting transfinite terms with infinitary $\Lambda\mu$-terms (I)

Transfinite terms that extend $\Lambda\mu$-terms:
Example of $\mu\alpha.(Y)\,\lambda f.f$ and $\mu\alpha.x$:



- $(Y)\,\lambda x\,.\,(f)x \sim (f)(f)\ldots$: subterm at limit ordinal not defined



Contrasting transfinite terms with infinitary $\Lambda\mu$-terms (II)

- Weak convergence with $\longrightarrow_{\mathsf{fst}}$
- Weak convergence with $\longrightarrow_{\beta} \cup \longrightarrow_{\beta_s} \cup \longrightarrow_{\mathsf{fst}}$? Consider $\mu\gamma.(\mu\alpha.x)\ \gamma$:

$$\mu\gamma.x \ {}_{\beta_s}\!\longleftarrow \mu\gamma.(\underbrace{\mu\alpha.x}_{A})\ \gamma \longrightarrow^{2}_{\mathsf{fst}} \lambda z\mu\gamma.(\underbrace{\lambda y\mu\alpha.x}_{B})\ [z \mid \gamma] \longrightarrow_{\beta} \lambda z\mu\gamma.(\underbrace{\mu\alpha.x}_{A})\ \gamma$$

  - A and B alternate in the reduction sequence
  - After $\longrightarrow_{\mathsf{fst}}$ reductions: $\lambda x_1 x_2 \ldots .(\lambda y_1 y_2 \ldots . t) x_1 x_2 \ldots$, β-redex at constant depth
  - Consider $\longrightarrow_{\mathsf{fst}}$ separately from $\longrightarrow_{\beta}$

Contrasting transfinite terms with infinitary $\Lambda\mu$-terms (III): Push down/Pull up

In $\Lambda\mu S$

- $\mu\alpha_0.x \longrightarrow_{\mathsf{fst}} \lambda x_0^\alpha.\mu\alpha_1.x \longrightarrow_{\mathsf{fst}} \dots \longrightarrow_{\mathsf{fst}}$
  $\lambda x_0^\alpha x_1^\alpha \dots x_n^\alpha.\mu\alpha_{n+1}.x \longrightarrow_{\mathsf{fst}} \dots$
  Push down: $\lambda x_0^\alpha x_1^\alpha \dots x_n^\alpha \dots .x$
- $\lambda x_1 x_2 \dots .(\lambda y_1 y_2 \dots .t) x_1 x_2 \dots$ could be pulled up to $\lambda x_1 x_2 \dots .t$.

In Ketema et al.

- In $\mu\alpha_0.x$, x is pushed down but $\mu\alpha$ stays: $\lambda x_0^\alpha x_1^\alpha \dots x_n^\alpha \dots .\mu\alpha.x$
- We would want $\lambda x_1 x_2 \dots .(\lambda y_1 y_2 \dots .t) x_1 x_2 \dots$ to become $\lambda x_1 x_2 \dots .t$ after ω steps: pull up t

Contrasting transfinite terms with infinitary $\Lambda\mu$-terms (IV)

- Study $\longrightarrow_{\mathsf{fst}}$ separately from β-reduction
- Find a more "trivial" topology than Ketema *et al.*
  - Consider infinitary terms: $\lambda x_1 \lambda x_2 \ldots . t$
  - Balls that do not only include prefixes of a term (non-Haussdorff)
- Conjecture on the limits via $\longrightarrow_{\mathsf{fst}}$
  - Example: $t = \mu\alpha\, \lambda y\, \mu\beta.(y)\alpha$ should converge to:
    $\lambda x^1_\alpha \lambda x^2_\alpha\ \ldots\ \lambda y\ \lambda x^1_\beta \lambda x^2_\beta \ldots .(y)[x^1_\alpha \mid x^2_\alpha \ldots]$

Perspectives

- Strong convergence for $\longrightarrow_{\mathsf{fst}}$
- Interaction between $\longrightarrow_{\beta}$ and $\longrightarrow_{\mathsf{fst}}$
- What we expect:
  - Understand properties: separability
  - Extend to more general calculi: stream hierarchy