Towards an atomic $\lambda\mu$-calculus

First year confirmation report

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Deep Inference [Guglielmi]

- Idea: apply inference rules at any depth
- Open deduction [Guglielmi, Gundersen, Parigot 2010]: composition of rules and formulas

$$\boxed{\begin{array}{c}A\\ \downarrow\Big\Vert\\ B\end{array}} \;:=\; A \;\Big|\; \boxed{r\,\dfrac{A}{B}} \;\Big|\; \boxed{\begin{array}{c}A_1\\ \downarrow\Big\Vert\\ A_2\end{array}} \wedge \boxed{\begin{array}{c}B_1\\ \downarrow\Big\Vert\\ B_2\end{array}} \;\Big|\; \boxed{\begin{array}{c}B_1\\ \uparrow\Big\Vert\\ A_1\end{array}} \to \boxed{\begin{array}{c}A_2\\ \downarrow\Big\Vert\\ B_2\end{array}} \;\Big|\; \begin{array}{c}\boxed{\begin{array}{c}A\\ \downarrow\Big\Vert\\ B\end{array}}\\ \cdots\cdots\\ \boxed{\begin{array}{c}B\\ \downarrow\Big\Vert\\ C\end{array}}\end{array}$$

Atomicity

- Medial rule:

$$m\,\frac{(A_1 \vee A_2) \to (B_1 \wedge B_2)}{(A_1 \to B_1) \wedge (A_2 \to B_2)}$$

- Atomic contraction:

$$\triangle\,\frac{A \to B}{(A \to B) \wedge (A \to B)} \quad \rightsquigarrow \quad \begin{array}{c} \left(\triangledown\,\dfrac{A}{A \vee A}\right) \to \left(\triangle\,\dfrac{B}{B \wedge B}\right) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ m\,\dfrac{(A \vee A) \to (B \wedge B)}{(A \to B) \wedge (A \to B)} \end{array}$$

The atomic λ-calculus Λ_a [1]

Λ_a is a refinement of Λ with:

- Linear Occurrences
- Atomic Duplication
- Natural sharing

[1] [Gundersen, Heijltjes, Parigot 2013]

The atomic λ-calculus Λ_a

$$\Lambda : \quad t, u ::= x \mid \lambda x.t \mid (t)u$$

- Intuitionistic Natural Deduction

$$\begin{array}{llll} \Lambda_a : t, u & ::= & x & \mid \quad \lambda x.t \\ & \mid & (t)u & \mid \quad u[c] \\ [c] & ::= & [x_1, \ldots, x_n \leftarrow t] & \\ & \mid & [x_1, \ldots, x_n \twoheadleftarrow \lambda y.t^n] & \\ t^n & ::= & \langle t_1, \ldots, t_n \rangle & \mid \quad t^n[c] \end{array}$$

- Intuitionistic Open Deduction

Idea behind the distributor:

$$\lambda y.t$$

$$\rightsquigarrow$$

$$\lambda y.\langle z_1, \ldots, z_n \rangle [z_1, \ldots, z_n \leftarrow t]$$

$$\rightsquigarrow_*$$

$$\lambda y.\langle t_1, \ldots, t_n \rangle$$

$$\rightsquigarrow$$

$$\lambda y_1.t_1 \qquad \ldots \quad \lambda y_n.t_n$$

Idea behind the distributor:

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t]$$

$$\rightsquigarrow$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n \rangle [z_1, \ldots, z_n \leftarrow t]]$$

$$\rightsquigarrow_*$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle t_1, \ldots, t_n \rangle [y_1, \ldots, y_n \leftarrow y]]$$

$$\rightsquigarrow$$

$$u\{\lambda y_1.t_1/x_1\} \ \ldots \ \{\lambda y_n.t_n/x_n\}$$

What about Classical Natural Deduction?

The $\lambda\mu$-calculus [Parigot 1992]

- Syntax:

$$\boxed{\lambda\mu : \quad t, u ::= \; x \mid \lambda x.t \mid (t)u \mid \mu\alpha.(t)\beta}$$

- $t \in \lambda\mu$: unnamed term; $(t)\beta$: named term
- Classical Natural Deduction $\cong_{CH} \lambda\mu$
- Reduction in $\lambda\mu$:

$$(\mu\alpha.n)u \longrightarrow_{\mu} \mu\alpha.\, n\{(w)u\alpha/(w)\alpha\}$$

Type system for the $\lambda\mu$-calculus

$$r\,\dfrac{t : \Gamma, A^x \vdash B \mid \Delta, D^\delta}{t' : \Gamma' \vdash B' \mid \Delta'}$$

Type system for the $\lambda\mu$-calculus

$$\mathit{Var}\ \frac{}{x : A^x \vdash A}$$

$$\lambda\ \frac{t : \Gamma, A^x \vdash B \mid \Delta}{\lambda x.t : \Gamma \vdash A \to B \mid \Delta}$$

$$@\ \frac{t : \Gamma \vdash A \to B \mid \Delta \quad u : \Gamma' \vdash A \mid \Delta'}{(t)u : \Gamma, \Gamma' \vdash B \mid \Delta, \Delta'}$$

$$@_n\ \frac{t : \Gamma \vdash A \mid \Delta}{(t)\alpha : \Gamma \vdash * \mid A^\alpha, \Delta}$$

$$\mu\ \frac{(t)\beta : \Gamma \vdash * \mid A^\alpha, \Delta}{\mu\alpha.(t)\beta : \Gamma \vdash A \mid \Delta}$$

$$\mathit{Lwk}\ \frac{t : \Gamma \vdash B \mid \Delta}{t : \Gamma, A^x \vdash B \mid \Delta}$$

$$\mathit{Rwk}\ \frac{t : \Gamma \vdash B \mid \Delta}{t : \Gamma \vdash B \mid \Delta, A^\alpha}$$

$$L\triangle\ \frac{t : \Gamma, A^x, A^x \vdash B \mid \Delta}{t : \Gamma, A^x \vdash B \mid \Delta}$$

$$R\triangle\ \frac{t : \Gamma \vdash B \mid \Delta, A^\alpha, A^\alpha}{t : \Gamma \vdash B \mid \Delta, A^\alpha}$$

Classical rules with the negation

We interpret $\neg A \equiv A \to \bot$, and consider the rules:

$$\frac{\Gamma, A \vdash * \mid \Delta}{\Gamma \vdash \neg A \mid \Delta}\ \neg_i$$

$$\frac{\dfrac{}{\neg A \vdash \neg A}\ Var \qquad \Gamma \vdash A \mid \Delta}{\Gamma, \neg A \vdash * \mid \Delta}\ \neg_e$$

Proving $\neg\neg A \to A$

$$\dfrac{\dfrac{\dfrac{}{\neg\neg A \vdash \neg\neg A}\,Var \qquad \dfrac{\dfrac{}{A \vdash A}\,Var}{\vdash \neg A \mid A}\,\neg_i}{\neg\neg A \vdash A}\,\neg_e}{\vdash \neg\neg A \to A}\,\to_i$$

Proving $\neg\neg A \to A$

$$
\dfrac{
  \dfrac{}{y : \neg\neg A^{y} \vdash \neg\neg A}\,\mathit{Var}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{x : A \vdash A}\,\mathit{Var}
        }{x : A \vdash A \mid \bot^{\delta}}\,\mathit{Rwk}
      }{(x)\alpha : A \vdash * \mid A^{\alpha}, \bot^{\delta}}\,@_{n}
    }{\mu\delta.(x)\alpha :\ A \vdash \bot \mid A^{\alpha}}\,\mu
  }{\lambda x.\mu\delta.(x)\alpha : \vdash \neg A \mid A^{\alpha}}\,\lambda
}{
  \dfrac{
    \dfrac{
      (y)\ \lambda x.\mu\delta.(x)\alpha : \neg\neg A \vdash \bot \mid A^{\alpha}
    }{((y)\ \lambda x.\mu\delta.(x)\alpha)\gamma : \neg\neg A \vdash * \mid A^{\alpha}, \bot^{\gamma}}\,@_{n}
  }{
    \dfrac{\mu\alpha.\ ((y)\ \lambda x.\mu\delta.(x)\alpha)\gamma : \neg\neg A \vdash A \mid \bot^{\gamma}}{\lambda y.\mu\alpha.\ ((y)\ \lambda x.\mu\delta.(x)\alpha)\gamma \vdash \neg\neg A \to A \mid \bot^{\gamma}}\,\lambda
  }\,\mu
}\,@
$$

Towards an atomic $\lambda\mu$-calculus

- Construct a $\lambda\mu$-calculus with explicit sharings
- Get the atomicity property: distributor for μ-abstractions

Open deduction with $\lambda\mu$

$$\boxed{\begin{array}{c} A \\ \downarrow\Vert \\ B \end{array}} \quad := \quad \cdots \quad \mid \quad \boxed{\begin{array}{c} A \\ \downarrow\Vert \\ B \end{array}} \vee \boxed{\begin{array}{c} C \\ \downarrow\Vert \\ D \end{array}} \quad \mid \quad \boxed{\begin{array}{c} A \\ \downarrow\Vert \\ B \end{array}} \mid \boxed{\begin{array}{c} C \\ \downarrow\Vert \\ D \end{array}}$$

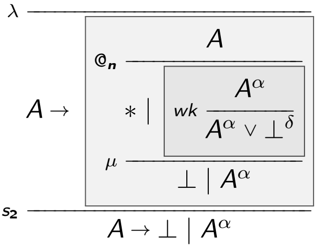
Open deduction rules for $\lambda\mu$

$$\lambda\ \frac{B}{A \rightarrow (B \wedge A)} \qquad @\ \frac{A \wedge (A \rightarrow B)}{B} \qquad \triangle\ \frac{A}{A \wedge \ldots \wedge A}$$

$$@_n\ \frac{A \mid \Delta}{* \mid A^{\alpha} \vee \Delta} \qquad \mu\ \frac{* \mid A^{\alpha} \vee \Delta}{A \mid \Delta} \qquad \nabla\ \frac{A \vee \ldots \vee A}{A}$$

$$s_1\ \frac{(A \mid \Delta) \wedge (B \mid \Delta')}{(A \wedge B) \mid \Delta \vee \Delta'} \qquad s_2\ \frac{A \rightarrow (B \mid \Delta)}{(A \rightarrow B) \mid \Delta}$$

Example for $A \vee \neg A$



Further work

Next steps:

- Atomicity with μ: medial rules
- Decompose reduction rules in $\lambda\mu$-calculus

Behaviour of the atomic $\lambda\mu$-calculus:

- PSN w.r.t. $\lambda\mu$
- $\lambda\mu$: confluent
- typed $\lambda\mu$: type preservation under reduction, strong normalization