A characterization of the Taylor expansion of λ-terms

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Recall on quantitative semantics

λ-calculus

$$\begin{array}{ccc} X & \longmapsto & (F)X \\ & & \big\downarrow_{\beta *} \\ & & Y \end{array}$$

Semantics

$$x \longmapsto f(x)$$

Recall on quantitative semantics

λ-calculus

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Semantics

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ & & \| \\ & & \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} (\partial_x^n f \cdot x^n) 0 \end{array}$$

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} (\partial_x^n f \cdot x^n) 0$ is the *Taylor Expansion* of f

λ-calculus $\xrightarrow{\text{Taylor Expansion}}$ Resource calculus

λ-calculus

Grammar: $\Lambda : T, U ::= x \mid \lambda x.T \mid (T)U$

$$(\lambda x.T)U \xrightarrow{\beta} T[U/x]$$

Resource calculus

Grammar: $\Delta : t, u ::= x \mid \lambda x.t \mid \langle t \rangle [u_1, \ldots, u_n]$

$$\langle \lambda x.t \rangle [u_1, \ldots, u_n] \xrightarrow{r} \sum_{\sigma \in \mathcal{S}_n} t\{u_{\sigma(1)}/x_1, \ldots, u_{\sigma(n)}/x_n\}$$

Substitutes each occurrence of x in t by only one u_i

Reduces to 0 otherwise

$$M \xmapsto{\text{Taylor Expansion}} \sum_{t \in \mathtt{taylor}(M)} \alpha_t t \xmapsto{\text{NF}} \text{NF}\left(\sum_{t \in \mathtt{taylor}(M)} \alpha_t t\right)$$

Goal: Characterize the image of this transformation

Theorem [Characterization]:

$\exists M$ λ-term s.t. $\sum_{t \in \Delta} \alpha_t \cdot t = \mathrm{NF}(taylor(M))$ iff

0 $\cdots$

1 $\cdots$

2 $\cdots$

3 $\cdots$

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0 Theorem [Ehrhard - Regnier]:
$\forall \alpha_t \in \mathrm{NF}(\mathrm{taylor}(M))$, if $\alpha_t \neq 0$ then $\alpha_t = \frac{1}{m(t)}$

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1 $\mathrm{FV}(\mathcal{T})$ is finite

2 $\mathcal{T}$ is r.e.

3 $\ldots$

Conditions 1 and 2: based on Barendregt's theorem

Theorem [Barendregt]:

Let $\mathcal{B}$ be a Böhm-like tree. There is a λ-term M such that $\mathrm{BT}(M) = \mathcal{B}$ if, and only if, $\mathrm{FV}(\mathcal{B})$ is finite and $\mathcal{B}$ is r.e..

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3 $\mathcal{T}$ is an ideal

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Resource calculus and Taylor expansion

Ideal

Two corollaries and further works

Plan

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Resource calculus

Grammar: $\Delta : t, u ::= x \mid \lambda x.t \mid \langle t\rangle[u_1, \ldots, u_n]$

Relation $\xrightarrow{r}$ (strongly normalizing, confluent):

$$\langle \lambda x.t\rangle[s_1, \ldots, s_n] \xrightarrow{r} \begin{cases} \{t\{s_{\sigma(1)}/x_1, \ldots, s_{\sigma(n)}/x_n\} \mid \sigma \in \mathcal{S}_n\} \\ \emptyset \text{ if } \deg_x(t) \neq n \end{cases}$$

Unique normal form: $\mathrm{NF}(t)$

$$\mathrm{NF}(\mathcal{T}) \triangleq \bigcup_{t \in \mathcal{T}} \mathrm{NF}(t)$$

Taylor expansion: $\Lambda \longrightarrow \mathcal{P}(\Delta)$

$\tau(x) \triangleq \{x\}$

$\tau(\lambda x.T) \triangleq \{\lambda x.t \mid t \in \tau(T)\}$

$\tau((T)U) \triangleq \{\langle t\rangle[u_1, \ldots, u_k] \mid t \in \tau(T); k \in \mathbb{N}; u_1, \ldots, u_k \in \tau(U)\}$

A first example : $\mathbf{S}$

$$\mathbf{S} := \lambda xyz.((x)z)(y)z$$

Böhm tree of $\mathbf{S}$:

$$\begin{array}{ccc} & \lambda xyz.x & \\ \diagup & & \diagdown \\ z & & y \\ & & \mid \\ & & z \end{array}$$

Taylor expansion of $\mathbf{S}$:

$$\tau(\mathbf{S}) = \{\lambda xyz.\langle x\rangle 1 1, \lambda xyz.\langle x\rangle \underbrace{[z,\ldots,z]}_{n}\,\underbrace{[\langle y\rangle 1,\ldots,\langle y\rangle 1]}_{m},\ldots\}$$

$$= \{\lambda xyz.\langle x\rangle[z^n][\langle y\rangle[z^{n_1}],\ldots,\langle y\rangle[z^{n_k}]]\ ;\ k,n,n_1,\ldots,n_k \in \mathbb{N}\}$$

$$= \mathrm{NF}(\tau(\mathbf{S}))$$

Two other examples

$$(\mathbf{S})\mathbf{II} = ((\lambda xyz.((x)z)(y)z)\lambda x.x)\lambda x.x \xrightarrow{\beta *} \lambda x.(x)x = \delta$$

$$\tau((\mathbf{S})\mathbf{II}) = \{\langle \lambda xyz.\langle x\rangle 1 1\rangle 1 1,$$
$$\langle \lambda xyz.\langle x\rangle [z,\dots,z][\langle y\rangle 1,\dots,\langle y\rangle 1]\rangle [I,\dots,I][I,\dots,I],\dots\}$$

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$$= \{\lambda x.\langle x\rangle [x^n], n \in \mathbb{N}\} = \tau(\delta)$$

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$$\mathbf{\Omega} = (\delta)\delta$$

$$\tau(\mathbf{\Omega}) = \{\langle \lambda x.\langle x\rangle [x^{n_0}]\rangle [\lambda x.\langle x\rangle [x^{n_1}],\ldots,\lambda x.\langle x\rangle [x^{n_k}]]; k, n_0,\ldots,n_k\in\mathbb{N}\}$$

$$\mathrm{NF}(\tau(\mathbf{\Omega})) = \emptyset$$

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Terms in normal form: $\Delta^{\mathrm{NF}} : t ::= \lambda x_0 \dots x_{m-1}.\langle y\rangle \mu_0 \dots \mu_{n-1}$

μ_i: finite multisets of simple terms in normal form

Uniform approximation $\preceq$

$\lambda x_0 \dots x_{m-1}.\langle y\rangle \mu_0 \dots \mu_{n-1} \preceq t$ iff

(i) $t = \lambda x_0 \dots x_{m-1}.\langle y\rangle \nu_0 \dots \nu_{n-1}$

(ii) $\forall i < n, |\mu_i| \neq \emptyset \implies \exists v \in |\nu_i|, \forall u \in |\mu_i|, u \preceq v$

$\preceq$-ideal

$\mathcal{T} \in \mathcal{P}(\Delta^{\mathrm{NF}})$ *ideal*: downward closed, directed

- $\tau(\mathbf{S}) = \{\lambda xyz.\langle x\rangle[z^n][\langle y\rangle[z^{n_1}], \dots, \langle y\rangle[z^{n_k}]] \; ; \; k, n, n_1, \dots, n_k \in \mathbb{N}\}$
- $\{\langle x\rangle[y, z]\}$
- $\{x[x]\}$,

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Two corollaries and further works

Corollary 1

Let $\mathcal{T} \in \mathcal{P}(\Delta^{\mathrm{NF}})$.
There is a normalizable λ-term M such that $\mathrm{NF}(\tau(M)) = \mathcal{T}$ iff

(i) $\mathrm{height}(\mathcal{T})$ is finite

(ii) $\mathcal{T}$ is a maximal clique

$\Delta^{\mathrm{NF}} : t \ ::= \lambda x_0 \ldots x_{m-1}.\langle y\rangle \mu_0 \ldots \mu_{n-1}$, μ_i finite multisets of simple terms in normal form.

Coherence $\frown$ on Δ^{NF}:

$\lambda x_0 \ldots x_{m-1}.\langle y\rangle \mu_0 \ldots \mu_{n-1} \frown t$ iff

(i) $t = \lambda x_0 \ldots x_{m-1}.\langle y\rangle \nu_0 \ldots \nu_{n-1}$

(ii) $\forall i < n, \forall u, u' \in |\mu_i \cdot \nu_i|, u \frown u'$

Clique: subset of a $\preceq$-ideal

$\mathcal{T} \in \mathcal{P}(\Delta^{\mathrm{NF}})$ *clique*: $\forall t, t' \in \mathcal{T}, t \frown t'$

Corollary 2

Let $\mathcal{T} \in \mathcal{P}(\Delta^{\mathrm{NF}})$.
There is a total λ-term M such that $\mathrm{NF}(\tau(M)) = \mathcal{T}$ iff

1. $\mathrm{FV}(\mathcal{T})$ is finite
2. $\mathcal{T}$ is r.e.
3. $\mathcal{T}$ is a maximal clique

Total terms

(i) $M \xrightarrow{h*} \lambda x_0 \dots x_{m-1}.(y)M_0 \dots M_{n-1}$

(ii) $M_0, \dots, M_{n-1}$ are total

Further works

Bring the results to more expressive calculi:

- $\Lambda\mu$-calculus
  - Cannot use Barendregt's theorem
- Non-Deterministic settings
  - Cannot use Ehrhard Regnier's theorem
- ...